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Excitation of internal waves in a stably-stratified atmosphere with considerable wind-shear

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The rate of generation of internal waves by a thin turbulent boundary layer was calculated in a previous paper for a stably-stratified atmosphere with no significant wind-shear outside the boundary layer by considering the excitation of normal modes of wave propagation. By using the concept of wave-packets propagating upwards from the boundary layer, the effects of wind-shear can be included. Conditions for the validity of the approximation are given. In general, the spectral distribution of wave-energy at a particular height takes large values in two bands of horizontal wave-number, one band deriving from wave-packets undergoing internal reflexion near that height and the other from wave-packets of very small local frequency that accumulate there. The 'reflexion' wave-numbers are dominant if the wind increases with height and the 'accumulation' wave-numbers if the wind initially decreases with height. The spectral energy distributions and intensities of the wave-motion are discussed in more detail for an atmosphere of uniform stability and unidirectional wind-shear. The accumulation process may lead to instability or overturning of the waves, and estimates are made of the probable scale and intensity of the 'clear-air' turbulence produced. An interesting point is that the rate of energy loss from the boundary layer by radiation of internal waves turns out to be comparable with the rate of production in the outer nine-tenths of the layer, both for atmospheric boundary layers and for the surface layer of the ocean. It seems likely that radiation limits the layer thickness to some extent.

1. Introduction

Over most of its depth, the earth's atmosphere is stably stratified and a parcel of air displaced vertically experiences a buoyancy force tending to restore it to its original level. One consequence is that internal gravity waves may propagate in it and transfer energy from one part of the atmosphere to another, a well-known example being the lee-waves produced by surface irregularities. Lee-waves are analogous to the bow-wave of a ship moving through water and are stationary with respect to time in a system moving with the obstacle that causes them. Here we are concerned with internal waves produced by moving sources, situated in a turbulent layer, waves which are analogous with the surface waves on water induced by the travelling pressure fluctuations under a turbulent boundary layer. Although the calculations and conclusions may be relevant to the generation of internal waves in the ocean or the upper atmosphere, the emphasis is on waves in the troposphere produced by the travelling disturbances in the earth's boundary layer and an object has been to discover the circumstances in which internal waves of this kind could give rise to travelling wave-clouds or to clear-air turbulence.

The problem has been simplified by assuming the atmosphere to be horizontally homogeneous and time-independent in its basic wind and temperature structure. In the real atmosphere, changes are frequent and the results can apply only for intervals over which the changes are comparatively small. Initially, the atmosphere is assumed to be free from internal waves but waves are induced by the boundary layer, represented by a travelling pattern of transient displacements at a particular level. The displacements are caused by vertical motions of the boundary layer and it is probable that the convection velocity of the pattern is nearly the wind velocity near the centre of the layer, say four-fifths of the wind at the top of the layer. With a knowledge of the power spectrum of the displacements, it is possible in principle to calculate the expected intensity and spectral distribution of the internal waves at any time, and the results should approximate to the added wave-motion in a real atmosphere that has changed little in the elapsed time.

In a recent paper (Townsend 1966), approximate expressions for the growth of internal gravity waves were derived for a stably-stratified atmosphere initially at rest, but the validity of the method depended on negligible variation of wind-velocity outside the boundary layer, a condition that is not particularly common in the real atmosphere. By using the wave-packet approximation, recently discussed by Lighthill (1965) and by Bretherton (1966), the limitation to negligible wind-shear can be avoided. The approximation considers the internal waves as an assembly of wave-packets of limited spatial extent which are radiated by the boundary layer, each propagating in fluid which is essentially uniform. By using refraction and energy conservation relations, trajectories and changes in energy and scale can be found and then the wave-motion at any height depends on the local characteristics and number-density of the wave-packets so far radiated. In the present context, the approximation is a good one if the scales of the imposed disturbances are small compared with the depth of the atmosphere, if the elapsed time is long compared with the Väisälä–Brunt frequency, and if the Richardson number is not small.

2. Waves generated by a travelling surface displacement

It is convenient to describe the motion with reference to a Cartesian co-ordinate system moving with the wind velocity just outside the boundary layer. z measures height above the boundary layer and the horizontal Ox axis is in the opposite direction to $\mathbf{V}$, the average convection velocity of the travelling disturbances relative to the co-ordinate system. The reason for the choice of Ox is that it allows positive vertical gradients of the horizontal wind component in the Ox direction to correspond with an increase with height of wind relative to the ground. The relevant properties of the undisturbed atmosphere are specified by the vertical distribution of the horizontal wind vector $\mathbf{U}(z)$, and the vertical distribution of the characteristic, Väisälä–Brunt frequency

$$\omega_1 = \left[-\frac{g}{\rho}\frac{d\rho}{dz}\right]^{\frac{1}{2}}.$$

By definition, $\mathbf{U}(0) = 0$.

Any surface disturbance may be analysed into travelling sinusoidal disturbances with vertical (complex) displacements of the form,†

$$\zeta = a_0(t)\exp i(\mathbf{k}.\mathbf{r}-\mathbf{k}.\mathbf{V}t),$$

where $a_0(t)$ is the complex amplitude, $\mathbf{k}$ is a two-dimensional wave-number vector with components l, m in the Ox, Oy directions, and $\mathbf{r} = (x, y)$ is the horizontal position vector. The convection velocity should be chosen so that the time variation of $a_0(t)$ is as small as possible for all realizations of the flow, and it may be a function of $\mathbf{k}$. It is assumed here that the convection velocity is the same for all relevant wave-numbers. Then $\mathbf{k}.\mathbf{V}$ is the wave-frequency in the co-ordinate system and $\omega(z) = \mathbf{k}.(\mathbf{V}-\mathbf{U})$ is the local wave-frequency as measured moving with the fluid at height z. For small vertical displacements, the system is linear and the basic problem is to find the wave-motion induced by a travelling sinusoidal disturbance in the given atmosphere. In the Boussinesq approximation, it may be shown from the inviscid equations of motion that travelling internal waves of small vertical displacement,

$$\zeta = \psi(z)\exp i(\mathbf{k}.\mathbf{r}-\mathbf{k}.\mathbf{V}t),$$

are possible if $\psi(z)$ satisfies

$$\psi'' + k^2\left(\frac{\omega_1^2}{\omega^2}-1\right)\psi - 2\omega^{-1}\mathbf{k}.\frac{d\mathbf{U}}{dz}\psi' - 2\omega^{-1}\mathbf{k}.\frac{d^2\mathbf{U}}{dz^2}\psi = 0, \tag{2.1}$$

and the proper boundary conditions. Equation (2.1) is identical in content with the Orr–Sommerfeld equation for a stably-stratified fluid.

If the Richardson number, $\omega_1^2(d\mathbf{U}/dz)^{-2}$, is not small and if ω/ω_1 does not vary greatly over height intervals of $\omega k^{-1}(\omega_1^2-\omega^2)^{-\frac{1}{2}}$, an approximate solution of (2.1) may be found by using the W.K.B. approximation. It is

$$\psi(z) = [\omega(\omega_1^2-\omega^2)]^{-\frac{1}{4}}\exp ik\int(\omega_1^2/\omega^2-1)^{\frac{1}{2}}dz. \tag{2.2}$$

Locally, the solution represents a transverse wave with wave-normals inclined to the horizontal at an angle of $\phi = \cos^{-1}\omega/\omega_1$ if $|\omega| < \omega_1$. It follows that a wave-packet of dimensions small compared with the scale of vertical inhomogeneity moves in a direction inclined to the horizontal by $\phi - \frac{1}{2}\pi$ with group-velocity,

$$G = (\omega_1/k)\sin\phi\cos\phi = (\omega/k)\sin\phi. \tag{2.3}$$

† Here and elsewhere, the physical displacements are the real parts of the complex displacements that appear in the equations.

The wave-motion emitted by the surface may be regarded as an assembly of wave-packets of horizontal wave-number $\mathbf{k}$ and local frequency $\mathbf{k}.(\mathbf{V}-\mathbf{U})$, and should fill the space below a front which propagates vertically upwards with the local value of $G\cos\phi$, at least until reflexion occurs. Near the surface, the wave amplitude is expected to be that of the surface disturbance and the upward flux of energy should depend only on conditions within the region of induced 'potential' flow.

To confirm the expected wave amplitude and to explore the approximation of a sharp wave-front, the method of the previous paper may be used to find the wave-motion produced in a uniformly-stratified, shear-free atmosphere of height H. 'Equation (2.4)' from Townsend (1965) may be written

$$\zeta(z,t) = a_0 \exp(-kz)\exp(i\mathbf{k}.\mathbf{V}t) + \sum_s a_s \psi_s(z)[\exp(-i\omega_s t) - \exp(i\mathbf{k}.\mathbf{V}t)], \quad (2.4)$$

where $\zeta(z,t)$ is the current amplitude. $\psi_s(z)$ is the vertical distribution function for the sth free mode, which satisfies (2.1) and the boundary conditions,

$$\left.\begin{aligned} &\psi(0) = \psi(H) = 0, \\ \text{i.e.}\quad &\psi_s(z) = \sin(s\pi z/H) \\ \text{and}\quad &\omega_s = \omega_1 kH(k^2H^2 + s^2\pi^2)^{-\frac{1}{2}}. \end{aligned}\right\} \quad (2.5)$$

From 'equation (2.5)' of the same paper,

$$\frac{a_s}{a_0} = \frac{2\pi s}{\omega_1^2 k^2 H^2}\frac{\omega_s^4}{\omega_s^2 - (\mathbf{k}.\mathbf{V})^2} \quad (2.6)$$

and, omitting the term $a_0\exp(-kz)\exp(i\mathbf{k}.\mathbf{V}t)$ which represents irrotational flow,

$$\zeta(z,t) = \frac{2\pi a_0}{\omega_1^2 k^2 H^2}\exp(-i\mathbf{k}.\mathbf{V}t)\sum_s \frac{s\omega_s^4}{\omega_s^2 - (\mathbf{k}.\mathbf{V})^2}\sin\left(s\frac{\pi z}{H}\right)[\exp\{-it(\omega_s - \mathbf{k}.\mathbf{V})\} - 1]. \quad (2.7)$$

The magnitude of the sum depends mostly on its largest terms, those for which $(\omega_s - \mathbf{k}.\mathbf{V})t$ is small. If both $\omega_1 t$ and $\mathbf{k}.\mathbf{V}t$ are large, the relation (2.5) between ω_s and s can be approximated by

$$\omega_s - \mathbf{k}.\mathbf{V} = -\frac{\pi^2 s_0}{\omega_1^2 k^2 H^2}(\mathbf{k}.\mathbf{V})^3(s - s_0), \quad (2.8)$$

where

$$s_0 = \frac{kH}{\pi\mathbf{k}.\mathbf{V}}(\omega_1^2 - (\mathbf{k}.\mathbf{V})^2)^{\frac{1}{2}}.$$

Using the approximation and replacing ω_s by $\mathbf{k}.\mathbf{V}$ except in factors involving their difference, we find

$$\zeta(z,t) = \frac{a_0}{\pi}\exp(-i\mathbf{k}.\mathbf{V}t)\sum_r \sin\left[(s_0 + r)\frac{\pi_2}{H}\right]\frac{\exp i\alpha r - 1}{-r}, \quad (2.9)$$

where

$$\alpha = (\mathbf{k}.\mathbf{V})^2\omega_1^{-2}[\omega_1^2 - (\mathbf{k}.\mathbf{V})^2]^{\frac{1}{2}}\pi t(kH)^{-1}.$$

All the approximations are valid if the factor $[\exp(-i\alpha r) - 1]/r$ becomes small for values of r less than s_0, i.e. if

$$\alpha s_0 = \mathbf{k}.\mathbf{V}t(\omega_1^2 - (\mathbf{k}.\mathbf{V})^2)\omega_1^{-2}$$

is large, a condition usually satisfied for large values of $\mathbf{k}.\mathbf{V}t$.

The expression for ζ is the sum of two parts,

$$\left.\begin{aligned}\zeta_1 &= \frac{a_0}{\pi}\exp\left[i\left(s_0\frac{\pi z}{H}-\mathbf{k}.\mathbf{V}t\right)\right]\sum_r\frac{\sin\frac{1}{2}\alpha r}{-r}\exp i\left\{r\left(\frac{\pi z}{H}+\tfrac{1}{2}\alpha\right)\right\}\\ \text{and}\quad \zeta_2 &= \frac{a_0}{\pi}\exp\left[i\left(-s_0\frac{\pi z}{H}-\mathbf{k}.\mathbf{V}t\right)\right]\sum_r\frac{\sin\frac{1}{2}\alpha r}{r}\exp -i\left\{r\left(\frac{\pi z}{H}-\tfrac{1}{2}\alpha\right)\right\}.\end{aligned}\right\} \tag{2.10}$$

Within the interval, $-\pi < x < \pi$,

$$\begin{aligned}\sum_{r=0}^{\infty}\frac{\sin rh}{r}\cos rx &= \tfrac{1}{2}\pi \quad \text{if} \quad |x| < h,\\ &= 0 \quad \text{if} \quad |x| > h,\end{aligned}$$

and so ζ_1 and ζ_2 have the non-zero values,

$$\left.\begin{aligned}\zeta_1 &= -a_0\exp i\left[s_0\frac{\pi z}{H}-\mathbf{k}.\mathbf{V}t\right] \quad \text{for} \quad 0 > z > -\frac{\alpha H}{\pi},\\ \zeta_2 &= a_0\exp i\left[-s_0\frac{\pi z}{H}-\mathbf{k}.\mathbf{V}t\right] \quad \text{for} \quad 0 < z < \frac{\alpha H}{\pi}.\end{aligned}\right\} \tag{2.11}$$

The period in z is $2H$ and the radiated waves, ζ_1 and ζ_2, spread respectively down and up from the levels, $z = \pm 2nH$, making their appearance in the physical atmosphere at the appropriate times as reflected waves. The vertical velocity of the wave-boundaries is

$$\frac{\alpha H}{\pi t} = \omega_1^{-2}k^{-1}(\mathbf{k}.\mathbf{V})^2[\omega_1^2-(\mathbf{k}.\mathbf{V})^2]^{\frac{1}{2}} = G\cos\phi. \tag{2.12}$$

For times less than $H/(G\cos\phi)$, the ζ_1 wave does not appear in the region $0 < z < H$ and the approximate solution is

$$\begin{aligned}\zeta &= a_0\exp i(-kz\tan\phi-\mathbf{k}.\mathbf{V}t) \quad \text{for} \quad z < Gt\cos\phi,\\ &= 0 \quad \text{for} \quad z > Gt\cos\phi.\end{aligned}$$

Sharp definition of the wave-front exists only if both s_0 and αs_0 are very large. The interpretation is that the front is not defined to better than the 'vertical' wavelength, $H/(\pi s_0) = k^{-1}\mathbf{k}.\mathbf{V}(\omega_1^2-(\mathbf{k}.\mathbf{V}^2)^{-\frac{1}{2}})$, which must be small compared with H and with the distance travelled. Equation (2.11) shows that the wave-amplitude just outside the boundary layer is the same as the amplitude of the surface displacement and, since the energy density of the wave-motion is $\omega_1^2\zeta\zeta^*$, the power radiated from the surface per unit area is $\omega_1^2\zeta\zeta^*G\cos\phi$. If $F_0(\mathbf{k})$ is the power-spectrum of the surface displacements, the power radiated is

$$W = \omega_1^2\int F_0(\mathbf{k})\,G(\mathbf{k})\cos\phi\,d\mathbf{k}. \tag{2.13}$$

Assuming the convection velocity to be the same for all wave-numbers, which is a reasonable approximation for the larger scale components of the boundary-layer motion, the power radiated is

$$W = \int k^{-1}(\mathbf{k}.\mathbf{V})^2[\omega_1^2-(\mathbf{k}.\mathbf{V})^2]^{\frac{1}{2}}F_0(\mathbf{k})\,d\mathbf{k}. \tag{2.14}$$

If the variations of wind and characteristic frequency are small within a layer of depth comparable with the scale of the displacements, the power radiated will be essentially the same as in the constant-stability, shear-free atmosphere. In an atmosphere with stability and wind varying with height, the amplitude of the wave-motion varies as $\omega^{-\frac{1}{2}}(\omega_1^2-\omega^2)^{-\frac{1}{4}}$ to the approximation of (2.2). The variation reflects the changes of group velocity with height and the interaction between the wave-motion and the wind-shear. Without approximation, a travelling wave $\psi(z)\exp i(\mathbf{k}.\mathbf{r}-\mathbf{k}.\mathbf{V}t)$ has a local energy density of $\frac{1}{2}\omega_1^2\psi\psi^*$, an upward flux of energy of

$$\overline{pw} = \tfrac{1}{4}k^{-2}\omega^3 i(\psi'\psi^*-\psi'^*\psi) \tag{2.15}$$

and a 'Reynolds stress' of

$$-\overline{uw} = -\tfrac{1}{4}k^{-1}\omega^2 i(\psi'\psi^*-\psi'^*\psi). \tag{2.16}$$

Conservation of energy requires that

$$\frac{d\overline{pw}}{dz} = -\overline{uw}\frac{dU}{dz}$$

and substituting from equations (2.15, 16), we find

$$i(\psi'\psi^*-\psi'^*\psi) \propto \omega^{-2}. \tag{2.17}$$

The same result may be obtained directly from equation (2.1). In the absence of reflexion, the upward flux of energy is proportional to the local frequency and the momentum flux is independent of height. If the direct and the various reflected waves are considered separately, the conclusions apply to the individual waves and the amplitudes vary as $\omega^{-\frac{1}{2}}(\omega_1^2-\omega^2)^{-\frac{1}{4}}$.

3. Spectral distribution of wave-amplitudes

Energy in the form of internal waves is radiated by Fourier components of the surface displacement such that $\mathbf{k}.\mathbf{V}$ is less than ω_0, the characteristic frequency just outside the boundary layer. For each Fourier component, a wave-front leaves the surface and propagates upwards at the local velocity $G\cos\phi$. Behind the front, the vertical amplitude is related to the surface amplitude by

$$aa^*\mathbf{k}.(\mathbf{V}-\mathbf{U})[\omega_1^2-(k.(\mathbf{V}-\mathbf{U}))^2]^{\frac{1}{2}} = a_0a_0^*(\mathbf{k}.\mathbf{V})[\omega_0^2-(\mathbf{k}.\mathbf{V})^2]^{\frac{1}{2}}, \tag{3.1}$$

where $a(\mathbf{k})$ is the vertical amplitude at the height where the wind-speed is $\mathbf{U}$ and the characteristic frequency is ω_1. To the wave-packet approximation, the amplitude becomes infinite where $\mathbf{k}.(\mathbf{V}-\mathbf{U}) = \omega$ becomes either zero or equal to the local characteristic frequency, and the vertical component of the group velocity is zero. Both cases are of special interest and it is worth considering in detail the behaviour of the wave-motion.

First, if ω/ω_1 exceeds one above a particular height z_0, equation (2.1) indicates an exponential variation of $\psi(z)$, typical of complete internal reflexion of the incident waves. If the variations of wind and density gradient are such that

$$k^2(1-\omega_1^2/\omega^2) = A^3(z-z_0),$$

then a solution of (2.1) for negligible U'' is

$$\psi(z) = \omega^{-1}\,\mathrm{Ai}\,[-A(z-z_0)], \tag{3.2}$$

where $\mathrm{Ai}(x)$ is an Airy integral with the asymptotic behaviour,

$$\mathrm{Ai}(x) \sim \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\sin\left(\tfrac{2}{3}(x)^{\frac{3}{2}}+\tfrac{1}{4}\pi\right) \text{ for large positive } x,$$
$$\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}(-x)^{\frac{3}{2}}\right) \text{ for large negative } x.$$

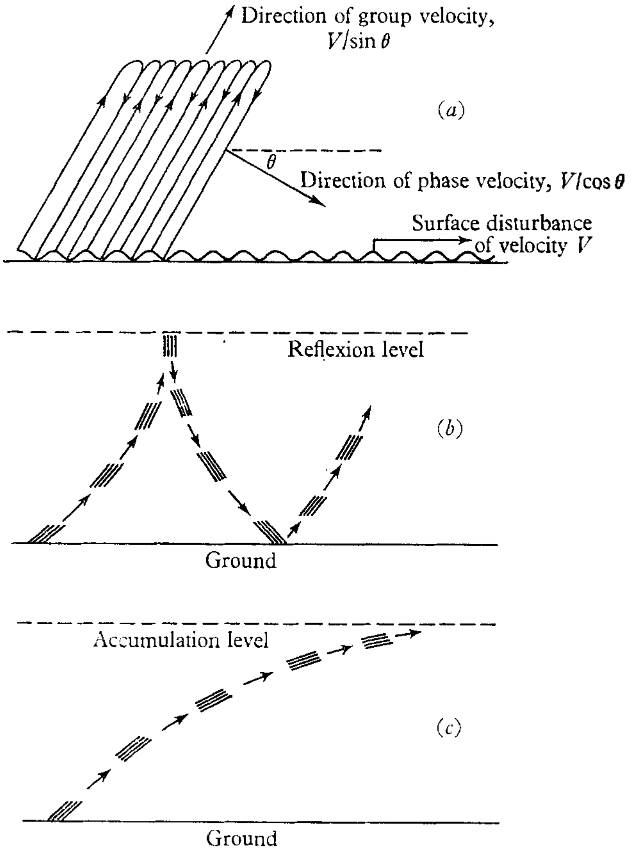

FIGURE 1. Propagation of internal waves in a stably stratified atmosphere. (*a*) Wave-propagation from a suddenly initiated sinusoidal disturbance at the ground. The parallel lines indicate the particle displacements. (*b*) Reflexion of a wave-packet at a level where $\omega = \omega_1$. (N.B. The indicated trajectory is a line parallel to direction of travel relative to the local fluid.) (*c*) Motion of a wave-packet approaching an accumulation level where its local frequency would be zero. (N.B. The wave-surfaces are diagrammatic and their separations are arbitrary.)

The solution represents a standing-wave pattern produced by waves undergoing reflexion near $z = z_0$, and the amplitudes are those given by (3.1) for $z_0 - z$ greater than A^{-1}. The amplitude becomes very small for $z - z_0$ greater than A^{-1} and the wave-packet description (which gives the intensity according to (3.1) for z less than z_0 and zero intensity for z greater than z_0) is satisfactory if details on the scale of A^{-1} are ignored. From the definition of A,

$$A^3 = -k^2\left[\frac{d^2\rho}{dz^2}\Big/\frac{d\rho}{dz} + 2\omega_1^{-1}\mathbf{k}\cdot\frac{d\mathbf{U}}{dz}\right] \tag{3.3}$$

and, for vertical gradients of wind comparable with ω_1, the scale A^{-1} is comparable with the horizontal wavelength k^{-1}. The reflexion is total and the reflected wave has the same amplitude as the incident wave at each level. Another feature of propagation near a level where $\omega = \omega_1$ is that the group velocity is proportional to the square-root of distance from the level, and the process of reflexion of a wave-packet takes only a finite time. In general, waves are reflected at the level where their local frequency becomes equal to the characteristic frequency, ω_1.

The other case of apparently infinite amplitude occurs if the local frequency becomes zero. Then the vertical wave-number is very large near the critical level and the wave-packet approximation should remain good, with the wave-front definable within a distance small compared with distance from the critical level. In contrast with the case of reflexion, the vertical group velocity varies as the square of distance from the level and the wave-front takes an infinite time to reach it. Moreover, the energy flux behind the wave-front is proportional to the local frequency, i.e. to distance from the critical level, and it seems certain that the waves cannot penetrate the level of zero local frequency in the approximation (see, however, Booker & Bretherton 1967). Where the waves have penetrated, the energy-density is nearly proportional to $\mathbf{k}.(\mathbf{V}-\mathbf{U})$ by equation (3.1) and wave-energy accumulates even though most of that radiated from the surface is lost by working against the gradient of horizontal velocity.

At any level, the spectrum of wave-energy is likely to be dominated by components undergoing *reflexion* or *accumulation* in neighbouring levels. In the wave-number plane, wave-numbers satisfying the reflexion condition,

$$\omega = \mathbf{k}.(\mathbf{V}-\mathbf{U}) = \omega_1,$$

lie on a line perpendicular to $\mathbf{U}-\mathbf{V}$, (the wind relative to the convection velocity), and distant $\omega_1/|\mathbf{V}-\mathbf{U}|$ from the origin.† Wave-numbers satisfying the accumulation condition, $\omega = 0$, lie on a line perpendicular to the relative wind and through the origin. In general, wave-energy is confined to Fourier components which have not satisfied either the condition for reflexion or that for accumulation at any lower level, i.e. not lying within the areas swept out by the two lines as the height increases, and it is most intense for components near the two lines.

Consider an atmosphere of uniform stability with the wind-vector $\mathbf{U}$ in the same direction at all heights and increasing steadily. Then the reflexion lines, $\mathbf{k}.(\mathbf{V}-\mathbf{U}) = \omega_1$, all pass through the wave-number $(\omega_1/V, -(\omega_1/V)\cot\delta)$ where δ is the angle between $\mathbf{V}$ and $-\mathbf{U}$. Figures 2 and 3 show the positions of the reflexion and accumulation lines for various heights: (i) for wind increasing with height ($\mathbf{U}.\mathbf{V}$ negative) and (ii) for wind initially decreasing with height ($\mathbf{U}.\mathbf{V}$ positive). All the energy radiated by the surface disturbance is in wave-numbers such that $\mathbf{k}.\mathbf{V} \leqslant \omega_1$, and the spectrum at any height is confined to the wave-numbers in the parallelogram bounded by the lines, $l = 0$ and ω_1/V, and by the accumulation and reflexion lines, $\mathbf{k}.(\mathbf{V}-\mathbf{U}) = 0$ and ω_1. Radiated energy outside the area has been removed by the processes of accumulation and reflexion. The intensities of Fourier components with wave-numbers near the reflexion

† Since a power spectrum must be an even function of $\mathbf{k}$, only the half-plane for $\mathbf{k}.\mathbf{V} > 0$ will be considered.

and accumulation lines are large, and the local wave-spectrum will be dominated by these components. If $\mathbf{U}.\mathbf{V}$ is negative, only the reflexion line passes through the wave-numbers of comparatively small k and moderate $\mathbf{k}.\mathbf{V}/\omega_1$ which are radiated most efficiently, and the centroid of the reflected wave-numbers is roughly in the wind-direction. The wave-crests then lie across the wind unless

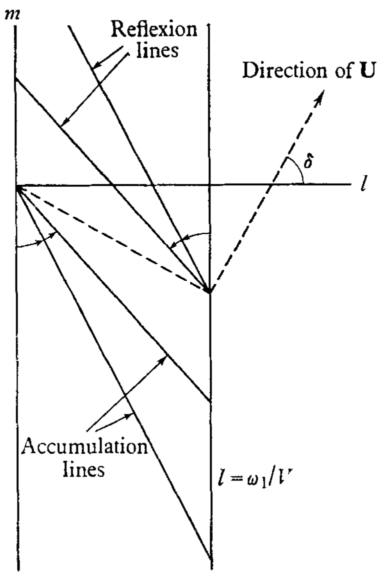

FIGURE 2

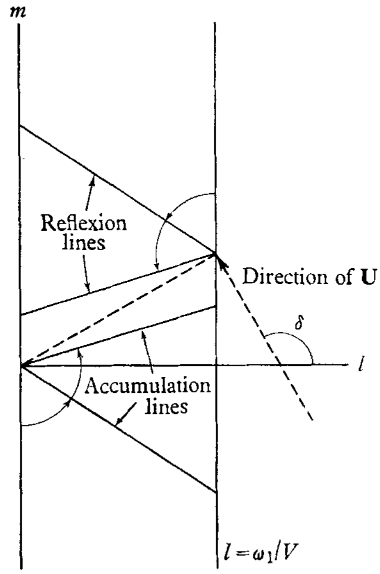

FIGURE 3

FIGURE 2. The horizontal wave-number plane, showing reflexion and accumulation lines for constant ω_1 and $\mathbf{U}.\mathbf{V} < 0$ ($\delta > \pi/2$).

FIGURE 3. The horizontal wave-number plane, showing reflexion and accumulation lines for constant ω_1 and $\mathbf{U}.\mathbf{V} > 0$ ($\frac{1}{2}\pi < \delta < \pi$).

$|\mathbf{U}| \gg |\mathbf{V}|$. If $\mathbf{U}.\mathbf{V}$ is positive, the accumulation line passes through the important wave-numbers and the centroid is at right angles to the relative wind. Then the wave-crests lie along the wind-direction. In both cases, if $|\mathbf{U}| \gg |\mathbf{V}|$, the parallelogram shrinks to a slit perpendicular to the wind direction but then the wave-energy is usually very small.

The discussion can be modified fairly easily if stability and wind vary in other ways. Reflexion and accumulation lines may be drawn to show the concentrations of wave-energy, but the possibility exists that the relevant Fourier components may have been stopped at lower levels.

4. Atmosphere with uniform wind-shear: reflected waves

Now consider the waves generated in an atmosphere with uniform density gradient and with uniform vertical gradient of wind, i.e. for which

$$\omega_1 = \text{constant}, \quad \mathbf{U} = \boldsymbol{\beta} z.$$

The waves radiated by Fourier components of horizontal wave-number $\mathbf{k}$ undergo reflexion at a height,

$$z_r = \frac{\omega_1 - \mathbf{k}.\mathbf{V}}{-\mathbf{k}.\boldsymbol{\beta}}, \tag{4.1}$$

which is positive only if $\mathbf{k}.\boldsymbol{\beta} < 0$. Waves for which $\mathbf{k}.\boldsymbol{\beta} > 0$ may accumulate near the level, $z_a = \mathbf{k}.\mathbf{V}/(\mathbf{k}.\boldsymbol{\beta})$ but are not reflected at any height. It is useful to consider separately the contributions to the wave-intensity from the two groups of wave-numbers.

Reflexion of the wave from a single Fourier component of the surface displacement will produce an interference pattern, extending downwards for a distance dependent on the coherence time of the emitted wave, but the separation of the maxima and minima is on too small a scale to be relevant to excitation by a continuous spectrum of displacement. So the effects of interference can be ignored and the total wave-intensity at a point obtained by summing the intensities of the initial wave and all the reflected waves that are passing at the time. At $t = 0$, waves begin to move upwards, each with the group velocity appropriate to its wave-number, $G\cos\phi$ given by equation (2.3), and the time necessary to reach a height z less than the reflexion height, $z_r = (\omega_1 - lV)/(-\mathbf{k}.\boldsymbol{\beta})$, is

$$t(z,\mathbf{k}) = \int_0^z \frac{dz}{G\cos\phi} = -\frac{k}{\mathbf{k}.\boldsymbol{\beta}}\left[\frac{(\omega_1^2 - l^2V^2)^{\frac{1}{2}}}{lV} - \frac{[\omega_1^2 - (lV - \boldsymbol{\beta}.\mathbf{k}z)^2]^{\frac{1}{2}}}{lV - \boldsymbol{\beta}.\mathbf{k}z}\right]. \tag{4.2}$$

The time to reach the reflexion height is

$$t_r(\mathbf{k}) = -\frac{k}{\mathbf{k}.\boldsymbol{\beta}}\frac{(\omega_1^2 - l^2V^2)^{\frac{1}{2}}}{lV}. \tag{4.3}$$

After reflexion, the wave travels down with the same velocity at each height as the upgoing wave, is reflected by the surface at time $2t_r$, and so on. It follows that the intensity at a height z less than z_r is 0 for t less than $t(z,\mathbf{k})$,

$$\begin{aligned} &\langle aa^*\rangle \quad \text{for} \quad t(z,\mathbf{k}) < t < 2t_r - t(z,\mathbf{k}),\\ &2\langle aa^*\rangle \quad \text{for} \quad 2t_r - t(z,\mathbf{k}) < t < 2t_r + t(z,\mathbf{k}),\\ &3\langle aa^*\rangle \quad \text{for} \quad 2t_r + t(z,\mathbf{k}) < t < 3t_r - t(z,\mathbf{k}),\end{aligned}$$

and so on, where

$$\langle aa^*\rangle = \frac{lV(\omega_1^2 - l^2V^2)^{\frac{1}{2}}}{(lV - \boldsymbol{\beta}.\mathbf{k}z)[\omega_1^2 - (lV - \boldsymbol{\beta}.\mathbf{k}z)^2]^{\frac{1}{2}}}\langle a_0 a_0^*\rangle$$

is the average intensity of the initial wave at height z. The discontinuous variation of intensity with time is inconvenient, but it may be approximated by the linear variation, $\langle aa^*\rangle\, t/t_r$, for large values of t/t_r.

Using the approximation of linear growth, the power spectrum of the wave displacements at height z due to reflexion components ($\mathbf{k}.\boldsymbol{\beta} < 0$) is

$$F(\mathbf{k}; z, t) = \frac{-\mathbf{k}.\boldsymbol{\beta} t l^2 V^2}{k(lV - \boldsymbol{\beta}.\mathbf{k}z)[\omega_1^2 - (lV - \boldsymbol{\beta}.\mathbf{k}z)^2]^{\frac{1}{2}}}F_0(\mathbf{k}) \tag{4.4}$$

and, knowing the power spectrum of the surface displacements, the wave-intensity is found by integrating over the triangular 'reflexion area' bounded by the lines, $l = 0$, $k.\boldsymbol{\beta} = 0$ and $lV - \mathbf{k}.\boldsymbol{\beta}z = \omega_1$. The form of the weighting factor implies that the dominant contributions are from wave-numbers near the reflexion line and that no large error arises if integration with respect to wave-number component in the direction of $\mathbf{V} - \mathbf{U}$ is between 0 and $\omega_1/|\mathbf{V} - \mathbf{U}|$ and if quantities other than $[\omega_1^2-(lV-\boldsymbol{\beta}.\mathbf{k}z)^2]^{\frac{1}{2}}$ are replaced by their values on the reflexion line. Naturally, it is necessary that $F_0(\mathbf{k})$ should be slowly varying near the reflexion line. Then,

$$\langle\zeta^2\rangle = \int_{-\cot(\delta-\epsilon)}^{\cot\epsilon} \frac{\pi\beta t V\sin\delta}{|V-\boldsymbol{\beta}z|^3}\frac{l^2V^2}{(1+x^2)^{\frac{1}{2}}}(\cot(\delta-\epsilon)+x)F_0(\mathbf{k})\,dx, \tag{4.5}$$

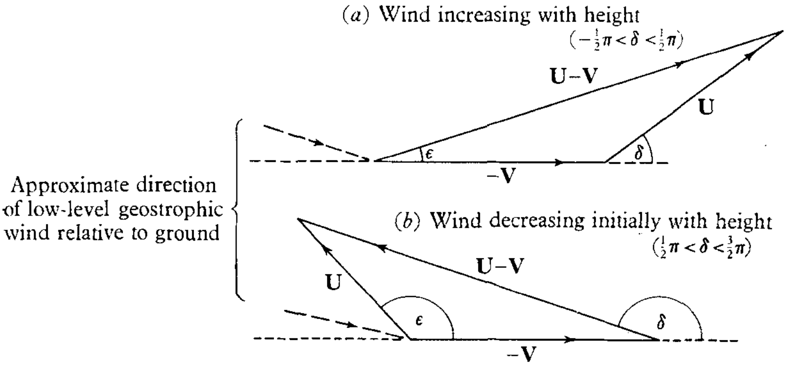

FIGURE 4. Vector diagrams for relative velocity with wind-shear.

where δ is the angle between $\mathbf{V}$ and $-\boldsymbol{\beta}$, ϵ is the angle between $\mathbf{V}$ and $\mathbf{V}-\mathbf{U}$ (figure 4), and $\mathbf{k}$ and l have the values on the reflexion line, i.e.

$$\mathbf{k} = (l,m) = k_1(\cos\epsilon - x\sin\epsilon, \sin\epsilon + x\cos\epsilon),$$

where $k_1 = \omega_1/|\mathbf{V}-\mathbf{U}|$. The variable x is the component of wave-number perpendicular to $\mathbf{V}-\mathbf{U}$ made non-dimensional with the scale $\omega_1/|\mathbf{V}-\mathbf{U}|$.

Outside a turbulent boundary layer in a neutrally stratified flow, the irrotational motion is known to be nearly statistically axisymmetric about directions normal to the surface, and it is probable that the power spectrum of the displacement velocities is isotropic in its two dimensions. Then it has the form,

$$f(\mathbf{k}) = k^2\phi_0(k), \tag{4.6}$$

and it is related to the power spectrum of the surface displacements by

$$f(\mathbf{k}) = [(\mathbf{k}.\mathbf{V})^2+\tau_1^{-2}]F_0(\mathbf{k}), \tag{4.7}$$

where $\tau_1^{-2} = -\left[\frac{d^2R(\tau)}{d\tau^2}\right]_{\tau=0}$ and $R(\tau) = \frac{\langle a_0(t)a_0^*(t+\tau)+a_0^*(t)a_0(t+\tau)\rangle}{2\langle a_0(t)a_0^*(t)\rangle}$.

Substituting in (4.5), the expected wave-intensity is

$$\langle\zeta^2\rangle = \frac{\pi\beta t\omega_1^2V\sin\delta}{|\mathbf{V}-\boldsymbol{\beta}z|^5}\int_{-\cot(\delta-\epsilon)}^{\cot\epsilon}\frac{l^2V^2\tau_1^2}{1+l^2V^2\tau_1^2}(1+x^2)^{\frac{1}{2}}(x+\cot(\delta-\epsilon))\phi_0(k)\,dx. \tag{4.8}$$

The dependence of wave-intensity on the motion in the boundary layer will be explored using the special form,

$$\phi_0(k) = \frac{16}{\pi^3}w_0^2L_0^4\exp\left(-\frac{4}{\pi}k^2L_0^2\right), \tag{4.9}$$

which describes motion with a root-mean-square velocity of w_0 and an integral scale of L_0, and assuming the auto-correlation time scale τ_1 to be the same for all wave-numbers. With this particular form for $\phi_0(k)$, the wave-intensity as a multiple of $\beta t L_0^2 w_0^2/V^2$ is

$$\frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2} = \frac{16}{\pi^2}\frac{\omega_1^2 L_0^2 V^3 \sin\delta}{|\mathbf{V}-\boldsymbol{\beta} z|^5}\int_{-\cot(\delta-\epsilon)}^{\cot\epsilon} \frac{l^2V^2\tau_1^2}{1+l^2V^2\tau_1^2}(1+x^2)^{\frac{1}{2}}[\cot(\delta-\epsilon)+x] \times \exp\left[-\frac{4}{\pi}\frac{\omega_1^2L_0^2}{|V-\boldsymbol{\beta} z|^2}(1+x^2)\right]dx, \quad (4.10)$$

i.e. a function of the non-dimensional height parameter, $\beta z/V$, and of the non-dimensional quantities, $\omega_1 L_0/V$, δ and $\omega_1\tau_1$. The magnitude of the wave-intensity can be estimated by considering some limiting cases. It will be assumed that $\omega_1 L_0/V$ and $\omega_1\tau_1$ are neither very large nor very small.

Consider the integral of (4.10) for large values of

$$\omega_1 L_0/(V\sin\epsilon) = \omega_1 L_0|\mathbf{V}-\boldsymbol{\beta} z|/(\beta z V\sin\delta),$$

a condition which may be satisfied if either $\sin\delta$ or $\beta z/V$ is small. If $\delta-\epsilon < \frac{1}{2}\pi$, the range of integration includes $x = 0$ but only positive values of x if $\delta-\epsilon > \frac{1}{2}\pi$. In both cases, the variation of the exponential factor ensures that significant contributions to the integral are confined to a range of x of extent comparable with $|\mathbf{V}-\boldsymbol{\beta} z|/(\omega_1 L_0)$. Since

$$l = \frac{\omega_1}{|\mathbf{V}-\boldsymbol{\beta} z|}(\cos\epsilon - x\sin\epsilon),$$

its corresponding variation is comparable with $\sin\epsilon/L_0$ and the variation of the factor $l^2V^2\tau_1^2/(1+l^2V^2\tau_1^2)$ is small if $\omega_1 L_0/(V\sin\epsilon)$ is large. Then the integral may be evaluated setting the factor equal to its value for the significant range of x, i.e. if $\delta-\epsilon < \frac{1}{2}\pi$, to its value for $x = 0$, $\omega_1^2\tau_1^2\cos^2\epsilon/(1+\omega_1^2\tau_1^2\cos^2\epsilon)$, and, if $\delta-\epsilon > \frac{1}{2}\pi$, to its value for $x = -\cot(\delta-\epsilon)$, $\omega_1^2\tau_1^2/(1+\omega_1^2\tau_1^2)$. Then,

$$\frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2} = \frac{16}{\pi^2}\frac{\omega_1^2 L_0^2 V^3\sin\delta}{|\mathbf{V}-\boldsymbol{\beta} z|^5}\frac{\omega_1^2\tau_1^2V^2\cos^2\epsilon}{|\mathbf{V}-\boldsymbol{\beta} z|^2+\omega_1^2\tau_1^2V^2\cos^2\epsilon}\int_{-\cot(\delta-\epsilon)}^{\cot\epsilon}(1+x^2)^{\frac{1}{2}} \times[\cot(\delta-\epsilon)+x]\exp\left[-\frac{4}{\pi}\frac{\omega_1^2L_0^2(1+x^2)}{|\mathbf{V}-\boldsymbol{\beta} z|^2}\right]dx. \quad (4.11)$$

Since $$\cot(\delta-\epsilon) = \cot\delta + \beta z/(V\sin\delta)$$
and $$\cot\epsilon = \cot\delta + V/(\beta z\sin\delta),$$

the range of integration is effectively infinite if δ is small and

$$\frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2} = \frac{4C}{\pi}\frac{V^2}{(V+\beta z)^2}\frac{\omega_1^2\tau_1^2V^2}{(V+\beta z)^2+\omega_1^2\tau_1^2V^2}\exp\left[-\frac{4}{\pi}\frac{\omega_1^2L_0^2}{(V+\beta z)^2}\right], \quad (4.12)$$

where $C = 1$ if $V+\beta z > \frac{1}{2}\omega_1 L_0$,
$C = 2\omega_1 L_0/(V+\beta z)$ if $V+\beta z < \frac{1}{2}\omega_1 L_0$.†

Roughly the wave-intensity falls off with height as $(V+\beta z)^{-4}$.

† Use has been made of the approximation that

$$\int_{-\infty}^{\infty}(1+x^2)^{\frac{1}{2}}\exp(-\tfrac{1}{2}a^2x^2)\,dx = 2a^{-2} \quad \text{for } a < (2/\pi)^{\frac{1}{2}}$$
$$= (2\pi)^{\frac{1}{2}}a^{-1} \quad \text{for } a > (2/\pi)^{\frac{1}{2}},$$

which has a maximum error of about 20 %.

If $\sin\delta$ is not small, the wave-intensity for small values of the non-dimensional height, $\beta z/V$, is given by

$$\frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2} = \frac{16}{\pi^2}\frac{\omega_1^2\tau_1^2}{1+\omega_1^2\tau_1^2}\frac{\omega_1^2 L_0^2}{V^2}\int_{-\cot\delta}^{\infty}(1+x^2)^{\frac{1}{2}}(\cos\delta + x\sin\delta) \times \exp\left[-\frac{4}{\pi}\frac{\omega_1^2 L_0^2}{V^2}(1+x^2)\right]dx. \quad (4.13)$$

Figure 5 shows in non-dimensional form the variations of low-level intensity for several values of $\omega_1 L_0/V$ as a function of δ, the angle between the convection velocity and the direction of wind-shear. They have been calculated from (4.13),

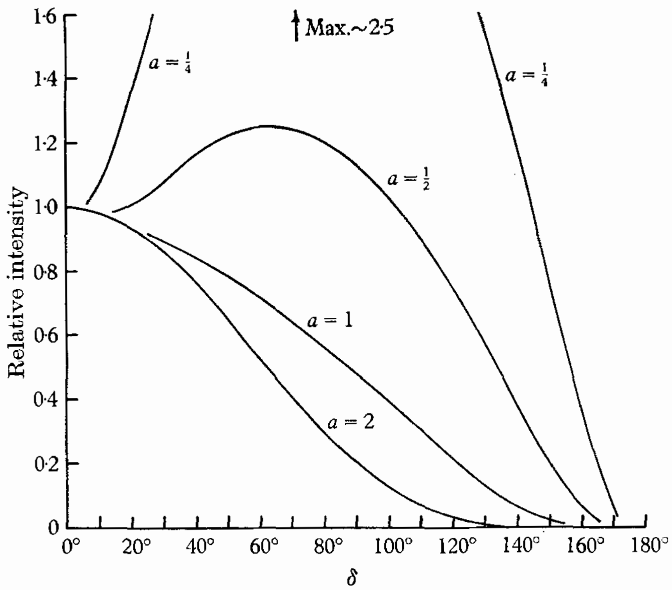

FIGURE 5. Relative variation of reflexion wave-intensity with δ for small values of $\beta z/V$, and for several values of $a = (8/\pi)^{\frac{1}{2}}\omega_1 L_0/V$. The absolute values for $\delta = 0$ are:

$a =$	$\frac{1}{4}$	$\frac{1}{2}$	1	2
$\frac{1}{2}\pi[1+(\omega_1\tau_1)^{-2}]\frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2} =$	2·00	1·98	1·77	0·68

using the approximation that $(1+x^2)^{\frac{1}{2}} = 1$ if $|x| < 1$, and $(1+x^2)^{\frac{1}{2}} = |x|$ if $|x| > 1$. Maximum intensity occurs for $\delta = 0$ if $\omega_1 L_0/V$ is large, and near $\delta = \frac{1}{2}\pi$ if $\omega_1 L_0/V$ is small.

For very large values of $\beta z/V$, $\delta - \epsilon$ is small and equation (4.10) reduces to

$$\frac{\langle\delta^2\rangle V^2}{\beta t w_0^2 L_0^2} = \frac{16}{\pi^2}\left(\frac{\beta z}{V}\right)^{-2}\frac{\omega_1^2 L_0^2}{V^2\sin^2\delta}\int_0^1 \frac{\omega_1^2\tau_1^2 u^2}{1+\omega_1^2\tau_1^2 u^2}u(1-u)\exp\left[-\frac{4}{\pi}\frac{\omega_1^2 L_0^2}{V^2\sin^2\delta}u^2\right]du. \quad (4.14)$$

If the factor $\omega_1^2\tau_1^2 u^2/(1+\omega_1^2\tau_1^2 u^2)$ is replaced by $[1-\exp(-\omega_1^2\tau_1^2 u^2)]$, the integral can be expressed in terms of the error integral and figure 6 shows the variation with δ for two values of $\omega_1\tau_1$. Zero intensity occurs both for $\delta = 0$ and for $\delta = \pi$. The result is not inconsistent with (4.12) which shows $\langle\zeta^2\rangle$ to vary as $(\beta z/V)^{-4}$ at large heights.

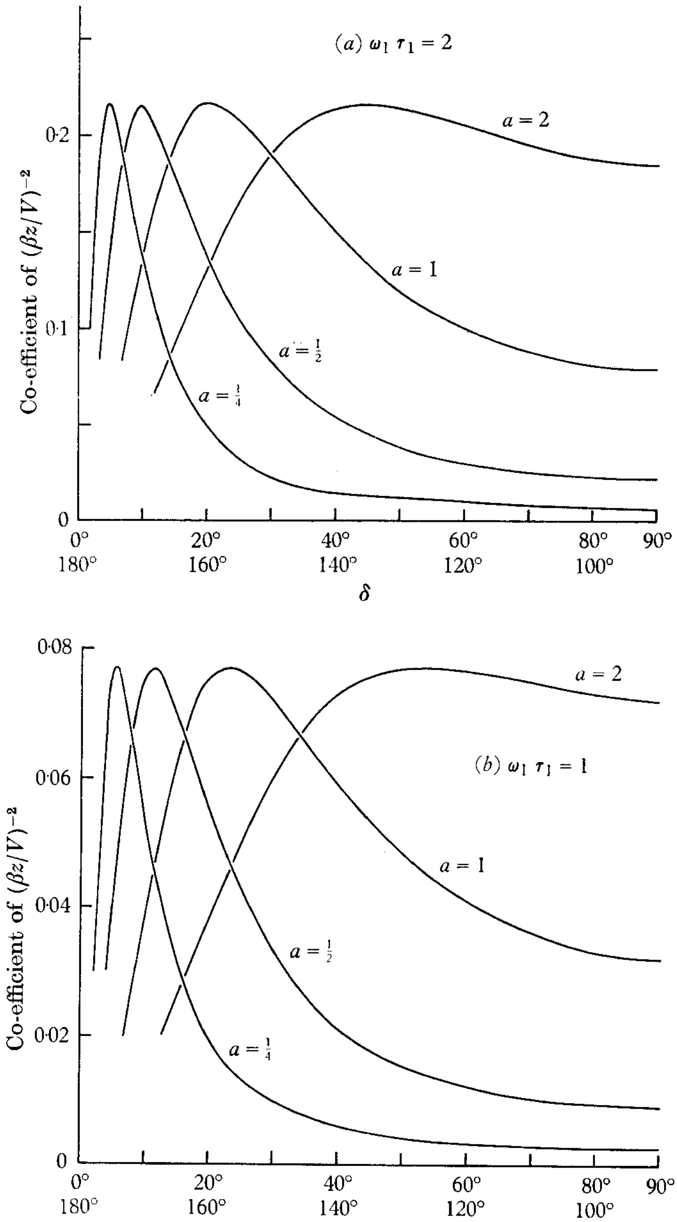

FIGURE 6. Variation of reflexion wave-intensity with δ for large values of $\beta z/V$, and for several values of $a = (8/\pi)^{\frac{1}{2}}\omega_1 L_0/V$. The ordinate is the value of

$$\frac{\pi}{2}\left(\frac{\beta z}{V}\right)^2 \frac{\langle\zeta^2\rangle V^2}{\beta t w_0^2 L_0^2}$$

(see equation (4.14)). (N.B. For large $\beta z/V$, the reflexion intensities for δ and $\pi-\delta$ are equal.)

5. Atmosphere with uniform wind-shear: accumulated waves

Components for which $\boldsymbol{\beta}.\mathbf{k}$ is positive are not reflected but accumulate near levels where their local frequencies would become zero. The time taken by the wave-front of a Fourier component of wave-number $\mathbf{k}$ to reach a height z is

$$t(z, \mathbf{k}) = \frac{k}{\mathbf{k}.\boldsymbol{\beta}}\left[\frac{\{\omega_1^2-(lV-\boldsymbol{\beta}.\mathbf{k}z)^2\}^{\frac{1}{2}}}{lV-\boldsymbol{\beta}.\mathbf{k}z} - \frac{(\omega_1^2-l^2V^2)^{\frac{1}{2}}}{lV}\right] \tag{5.1}$$

and becomes infinite at the accumulation level, $z_a = lV/(\boldsymbol{\beta}.\mathbf{k})$. The component of wave-number perpendicular to the accumulation line is

$$m' = (lV-\boldsymbol{\beta}.\mathbf{k}z)/|\mathbf{V}-\boldsymbol{\beta}z|,$$

and, for large values of βt, very nearly

$$t(z, \mathbf{k}) = \frac{k\omega_1}{\mathbf{k}.\boldsymbol{\beta}m'|\mathbf{V}-\boldsymbol{\beta}z|}.$$

It follows that only those components with wave-numbers such that $m' > m_c$ can reach the height z in time t, where

$$m_c = \frac{k\omega_1}{\boldsymbol{\beta}.\mathbf{k}|\mathbf{V}-\boldsymbol{\beta}z|}t^{-1}. \tag{5.2}$$

To the wave-packet approximation, the components that are present have intensities that are independent of time and given by (3.1), and so the rate of increase of wave-intensity by accumulation is

$$\frac{d}{dt}\langle\zeta^2\rangle = -\int F_0(\mathbf{k})\frac{lV(\omega_1^2-l^2V^2)^{\frac{1}{2}}}{(lV-\boldsymbol{\beta}.\mathbf{k}z)[\omega_1^2-(lV-\boldsymbol{\beta}.\mathbf{k}z)^2]^{\frac{1}{2}}}\frac{dm_c}{dt}dl'. \tag{5.3}$$

Here l' is the component of wave-number along the accumulation line and the integration is along the boundary, $m' = m_c$. The boundary is very close to the accumulation line for large βt, and then

$$\frac{d}{dt}\langle\zeta^2\rangle = t^{-1}\int F_0(\mathbf{k})\frac{lV(\omega_1^2-l^2V^2)^{\frac{1}{2}}}{\omega_1|\mathbf{V}-\boldsymbol{\beta}z|}dl', \tag{5.4}$$

the integration now being along the accumulation line. Substituting the isotropic form for the velocity spectrum function, and with a change of variable to $u = l'V\sin\epsilon/\omega_1 = l/\omega_1$,

$$\frac{V^2}{w_0^2L_0^2}\frac{d}{dt}\langle\zeta^2\rangle = \frac{32}{\pi^3}\frac{\omega_1^2L_0^2t^{-1}}{V|\mathbf{V}-\boldsymbol{\beta}z|\sin^3\epsilon}\int_0^1\frac{\omega_1^2\tau_1^2u^2}{1+\omega_1^2\tau_1^2u^2}u(1-u^2)^{\frac{1}{2}} \times\exp\left[-\frac{4}{\pi}\frac{\omega_1^2L_0^2}{V^2\sin^2\epsilon}u^2\right]du, \tag{5.5}$$

showing that the intensity increases nearly as $\log\beta t$.

Approximating the function $\omega_1^2\tau_1^2u^2/(1+\omega_1^2\tau_1^2u^2)$ by $(1-\exp(-\omega_1^2\tau_1^2u^2))$, the integral can be expressed in terms of the tabulated function

$$\int_0^t \exp x^2 dx.$$

Figure 7 shows the intensity factor, $P(a)$, defined so that

$$\frac{V^2}{w_0^2L_0^2}\frac{d}{dt}\langle\zeta^2\rangle = \frac{4}{\pi^2}\frac{V}{\beta z\sin\delta}t^{-1}P\left(\frac{2}{\pi^{\frac{1}{2}}}\frac{\omega_1L_0}{V\sin\epsilon}\right) \tag{5.6}$$

for three values of $\omega_1\tau_1$, 1, 2 and ∞. For the interesting case of small $\pi-\delta$, large rates of accumulation are confined to values of $\sin\epsilon$ between $5\omega_1 L_0/V$ and $0{\cdot}5\omega_1 L_0/V$ with $\beta z/V$ nearly one.†

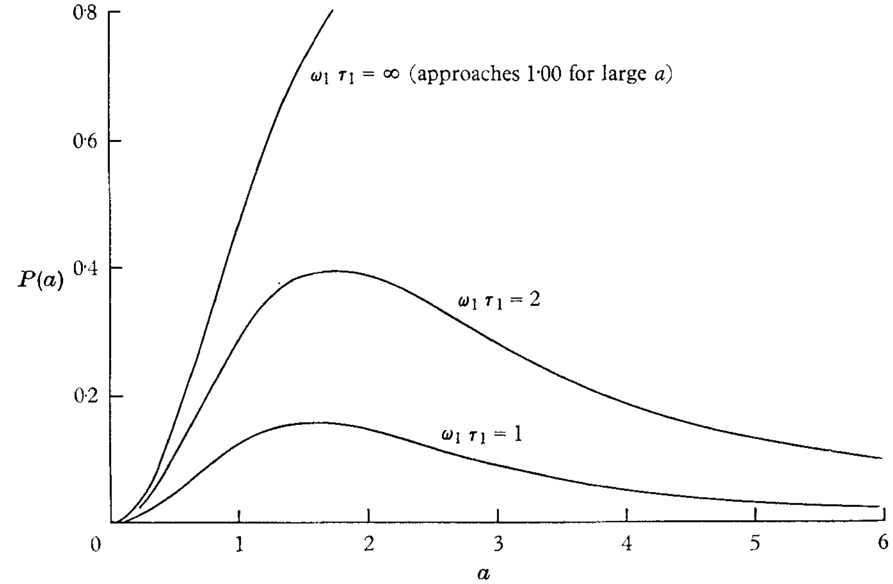

FIGURE 7. Variation of accumulation wave-intensity, expressed in terms of the function $P(a)$ defined in (5.6).

Near an accumulation level, the vertical component of wave-number for the packet becomes very large and it is expected that the mean square of the vertical gradient of displacement, $\partial\zeta/\partial z$, will increase with time more rapidly than the mean-square displacement. Large values of the vertical gradient are favourable to the development of instabilities in the fluid and it is useful to have an estimate of the rate of growth. The power spectrum of $\partial\zeta/\partial z$ is obtained by multiplying the power spectrum of the displacement by $(k\tan\phi)^2$, where

$$\cos\phi = \omega/\omega_1 = (lV-\boldsymbol{\beta}.\mathbf{k}z)/\omega_1.$$

Using the same argument as before, the rate of increase of the mean-square gradient is

$$\frac{d}{dt}\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = t\int F_0(\mathbf{k})\,\frac{(\boldsymbol{\beta}.\mathbf{k})^2 lV(\omega_1^2-l^2V^2)^{\frac{1}{2}}}{\omega_1|\mathbf{V}-\boldsymbol{\beta}z|}\,dl', \tag{5.7}$$

and substituting the isotropic form for the spectrum function leads to

$$\frac{d}{dt}\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{32}{\pi^3}\beta^2 t\frac{\omega_1'' L_0''}{V^4}\,\frac{w_0^2}{V|\mathbf{V}-\boldsymbol{\beta}z|}\,\frac{\sin^2(\delta-\epsilon)}{\sin^5\epsilon}\int_6^1 \frac{\omega_1^2\tau_1^2u^2}{1+\omega_1^2\tau_1^2u^2}u^3(1-u^2)^{\frac{1}{2}}$$
$$\times\exp\left[-\frac{4}{\pi}\frac{\omega_1^2L_0^2u^2}{V^2\sin^2\epsilon}\right]du. \tag{5.8}$$

† For small $\beta z/V$, $a \approx (2/\pi^{\frac{1}{2}})[\omega_1 L_0/(\beta z\sin\delta)]$ and $P(a)$ is exponentially small for large values of a. The accumulation rate is so very small in spite of the factor $V/(\beta z\sin\delta)$ in equation (5.6).

With the same approximation for $\omega_1^2\tau_1^2u^2/(1+\omega_1^2\tau_1^2u^2)$, the integral can be evaluated in terms of

$$\int_0^t \exp x^2 dx,$$

and figure 8 shows the intensity factor $Q(a)$, defined so that

$$\frac{d}{dt}\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{3}{2\pi}\beta^2 t\frac{w_0^2}{V^2}\frac{V^3\sin\delta}{\beta z|\mathbf{V}-\boldsymbol{\beta}z|^2}Q\left(\frac{2}{\pi^{\frac{1}{2}}}\frac{\omega_1 L_0}{V\sin\epsilon}\right) \tag{5.9}$$

for the same three values of $\omega_1\tau_1$. The rate of increase is large in conditions where the increase of wave-intensity is large, but the mean-square gradient increases as the square of the time.

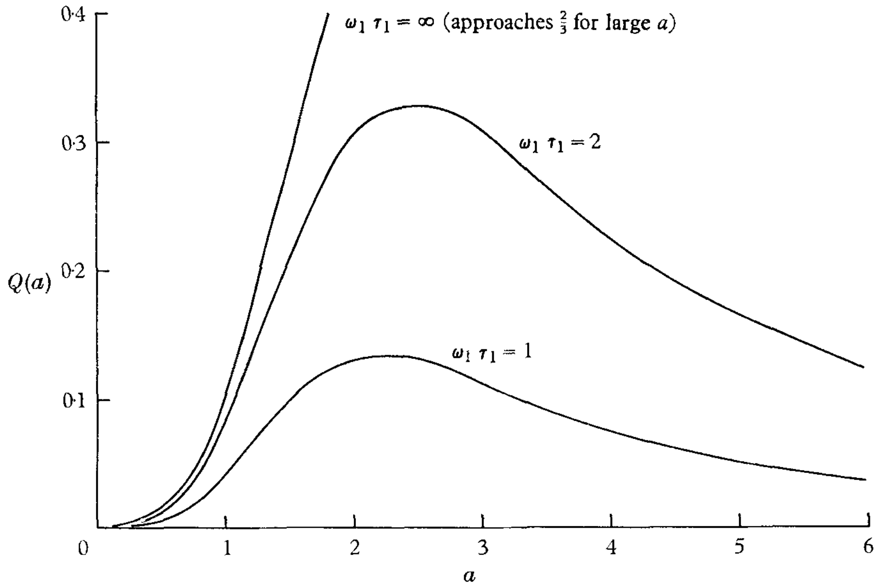

FIGURE 8. Variation of vertical gradients of wave displacement, expressed in terms of the function $Q(a)$ defined in (5.9).

6. Production of turbulence by the accumulation process

If the amplitude of the internal waves is large, the simple linear theory is inadequate and any of a number of non-linear effects may occur. Some of these are conservative of energy, transferring it from one wave-number to another, but some resemble hydrodynamic instabilities and may grow and develop into dissipative turbulent motion. With internal waves of large amplitude, two kinds of instability appear to be possible: (i) convective instability developing in regions where wave-displacements have reversed the vertical gradient of density. The condition for reversal is that

$$\frac{\partial\zeta}{\partial z} < -1.$$

(ii) Shear instability developing in regions of high rate of shear. For a sinusoidal internal wave, maximum velocity gradient coincides with zero displacement and the stability criterion depends on the velocity gradient of the waves and on the undisturbed density gradient of the atmosphere. For inclined flow, the energy argument used in the 'just no turbulence' criterion of L. F. Richardson would show that the flow becomes unstable if the Richardson number formed from components of gravity and density gradient in the direction of velocity shear is less than a critical value, i.e.

$$\mathrm{Ri} = -\frac{g}{\rho}\frac{d\rho}{dz}\sin^2\theta/(\mathrm{grad}\,u)^2 = \frac{\omega_1^2\sin^2\theta}{(\mathrm{grad}\,u)^2} < (\mathrm{Ri})_c. \tag{6.1}$$

In either case, turbulence is unlikely to develop unless the unstable configuration persists for times long compared with characteristic times of the flow.

An internal wave component of horizontal wave-number k has its wave-normal inclined to the horizontal at an angle $\theta = \cos^{-1}\omega/\omega_1$. If a is the amplitude of vertical displacement and ω is the local frequency, the velocity amplitude at right angles to the wave-normal is $\omega a/\cos\theta = \omega_1 a$, and, since the three-dimensional wave-number is $k/\cos\theta$, the amplitude of the velocity gradient is $k\omega_1^2 a/\omega$. The components of gravity and density gradient along the wave-normal are $g\sin\theta$ and $d\rho/dz\sin\theta$, so that the Richardson number formed from the undisturbed density gradient and the rate of shear in the wave is

$$\mathrm{Ri}(k) = \frac{\omega^2(\omega_1^2-\omega^2)}{k^2\omega_1^4 aa^*}. \tag{6.2}$$

An effective Richardson number for the whole spectrum of waves is then

$$\mathrm{Ri} = \left[\int \frac{k^2\omega_1^4}{\omega^2(\omega_1^2-\omega^2)} F(\mathbf{k})\,d\mathbf{k}\right]^{-1}. \tag{6.3}$$

Since the vertical component of wave-number is $k(\omega_1^2-\omega^2)^{\frac{1}{2}}/\omega$, the power spectrum of $\partial\zeta/\partial z$ is

$$P(\mathbf{k}) = k^2(\omega_1^2-\omega^2)/\omega^2 F(k)$$

and the effective Richardson number may be expressed as

$$\mathrm{Ri} = \left[\int \frac{(\omega_1^2-\omega^2)^2}{\omega_1^4} P(\mathbf{k})\,d\mathbf{k}\right]^{-1}. \tag{6.4}$$

Large values of $\partial\zeta/\partial z$ are likely to arise only through accumulation of components whose local frequencies are small compared with ω_1, and the effective Richardson number is very nearly

$$\mathrm{Ri} = \left[\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle\right]^{-1}. \tag{6.5}$$

The condition for shear instability of the wave system is nearly the same in form as the condition for convective instability, and it is possible that both forms of instability occur together. Since the conditions are likely to be met only by the accumulation process, the duration of locally unstable conditions, of order ω^{-1}, is long compared with ω_1^{-1} and with $|\mathrm{grad}\,\mathbf{u}|^{-1}$, the two relevant time scales.

Consider now the steady state that may be expected when energy loss to the turbulent motion prevents further intensification of the wave-motion. If the action of the turbulence on the waves is equivalent to that of an eddy viscosity $\nu_T(z)$, each Fourier component has its energy reduced by a factor of

$$\exp - \int_0 \frac{\nu_T k^2 \omega_1^2}{\omega^2 G \cos\phi} dz = \exp - \int_0 \frac{\nu_T k^3 \omega_1''}{\omega^4(\omega_1^2 - \omega^2)^{\frac{1}{2}}} dz$$

(Townsend 1965). For components near their accumulation level, wind-shear and eddy viscosity vary slowly with height compared with the local frequency, $\omega = lV - \boldsymbol{\beta}.\mathbf{k}z$, and

$$\int_0 \frac{\nu_T k^3 \omega_1^4}{\omega^4(\omega_1^2 - \omega^2)^{\frac{1}{2}}} dz = \tfrac{1}{3}\nu_T \frac{k^3 \omega_1^3}{\boldsymbol{\beta}.\mathbf{k}\omega^3}. \tag{6.6}$$

It follows that the component is absorbed almost entirely during its passage through a shallow layer centred around the level where $\frac{1}{3}\nu_T k^3 \omega_1^3 (\boldsymbol{\beta}.\mathbf{k}\omega^3)^{-1} = 1$, and that the wave-motion at height z is derived almost entirely from components for which

$$\omega > (3\boldsymbol{\beta}.\mathbf{k}/\nu_T)^{-\frac{1}{3}} k\omega_1.$$

The magnitude of $\langle(\partial\zeta/\partial z)^2\rangle$ is found by integrating $G(\mathbf{k})$ over all values of $\mathbf{k}$ satisfying this condition. Since the power spectrum varies as m'^{-3} near the accumulation line, where $m' = \omega/|\mathbf{V} - \boldsymbol{\beta}z|$, the integration with respect to m' can be done neglecting variations of other factors and then

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \int_0^{\omega_1/(V\sin\epsilon)} F_0(\mathbf{k}) \frac{lV(\omega_1^2 - l^2V^2)^{\frac{1}{2}}}{\omega_1|\mathbf{V} - \boldsymbol{\beta}z|} \left(\frac{3\boldsymbol{\beta}.\mathbf{k}}{\nu_T}\right)^{\frac{2}{3}} dl', \tag{6.7}$$

the integration being along the accumulation line. Comparison with (5.7) shows that the weighting factor of $F_0(\mathbf{k})$ in the integral differs only by a factor of $[\frac{1}{3}\nu_T(\boldsymbol{\beta}.\mathbf{k})^2]^{-\frac{2}{3}} t^{-1}$, and so

$$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle = \frac{3}{2\pi}\beta^2[\tfrac{1}{3}\nu_T(\boldsymbol{\beta}.\mathbf{k})_q^2]^{-\frac{2}{3}} \frac{w_0^2}{V^2} \frac{V^3 \sin\delta}{\beta z|\mathbf{V} - \boldsymbol{\beta}z|^2} Q\left(\frac{2}{\pi^{\frac{1}{2}}} \frac{\omega_1 L_0}{V\sin\epsilon}\right), \tag{6.8}$$

where $Q(a)$ is the function plotted in figure 8 and $(\boldsymbol{\beta}.\mathbf{k})_q$ is an average value along the accumulation line. Since $(\boldsymbol{\beta}.\mathbf{k}) = \beta l' \sin(\delta - \epsilon)$, we require an effective value of l'. If $\omega_1 L_0/(V\sin\epsilon)$ is small, the whole range of l' is involved and

$$l'_m \approx \tfrac{1}{2}\omega_1/(V\sin\epsilon).$$

If $\omega_1 L_0/(V\sin\epsilon)$ is large, contributions for large $l'L_0$ are negligible and so $l'_m \approx L_0^{-1}$. Roughly, then,

$$(\boldsymbol{\beta}.\mathbf{k})_q = \left[1 - \exp\left(-\frac{1}{2}\frac{\omega_1 L_0}{V\sin\epsilon}\right)\right]\frac{\beta}{L_0}\sin(\delta - \epsilon). \tag{6.9}$$

If the production of turbulence prevents the occurrence of values of $\langle(\partial\zeta/\partial z)^2\rangle$ larger than one, (6.8) gives the eddy viscosity as

$$\nu_T = \frac{3\beta L_0^2 \sin^{-2}(\delta - \epsilon)}{\left[1 - \exp\left(-\frac{1}{2}\frac{\omega_1 L_0}{V\sin\epsilon}\right)\right]^2}\left[\frac{3}{2\pi}\frac{w_0^2}{V^2}\frac{V^3\sin\delta}{\beta z|\mathbf{V} - \boldsymbol{\beta}z|^2} Q\left(\frac{2}{\pi^{\frac{1}{2}}}\frac{\omega_1 L_0}{V\sin\epsilon}\right)\right]^{\frac{3}{2}}. \tag{6.10}$$

The eddy viscosity can be used to calculate the rate of energy transfer to the turbulent motion. It is

$$\bar{\epsilon} = \int \nu_T k^2 \frac{\omega_1^2}{\omega^2} \omega_1^2 \frac{lV(\omega_1^2 - l^2V^2)^{\frac{1}{2}}}{\omega(\omega_1^2-\omega^2)^{\frac{1}{2}}} F_0(\mathbf{k})\, d\mathbf{k} \tag{6.11}$$

and again the integrand varies as m'^{-3} near the accumulation line. Integrating with respect to m',

$$\bar{\epsilon} = \int_0^{\omega_1/(V \sin \epsilon)} \nu_T \frac{\omega_1 l V(\omega_1^2 - l^2 V^2)^{\frac{1}{2}}}{|\mathbf{V} - \boldsymbol{\beta} z|} \left(\frac{3(\boldsymbol{\beta} \, . \, \mathbf{k})}{\nu_T}\right)^{\frac{2}{3}} F_0(\mathbf{k})\, dl', \tag{6.12}$$

in which the integrand differs from that of (5.4) only by a factor of

$$(9\nu_T(\boldsymbol{\beta} \, . \, \mathbf{k})^2)^{-\frac{1}{3}} \omega_1^{-2} t^{-1}.$$

Then

$$\bar{\epsilon} = \frac{4}{\pi^2} \omega_1^2 L_0^2 [9\nu_T(\boldsymbol{\beta} \, . \, \mathbf{k})_p^2]^{\frac{1}{3}} \frac{w_0^2}{V^2} \frac{V}{\beta z \sin \delta} P\left(\frac{2}{\pi^{\frac{1}{2}}} \frac{\omega_1 L_0}{V \sin \epsilon}\right), \tag{6.13}$$

where $P(a)$ is the function plotted in figure 7 and $(\boldsymbol{\beta} \, . \, \mathbf{k})_p$ is another average along the accumulation line. Substituting for the eddy viscosity from (6.10),

$$\bar{\epsilon} = \left(\frac{216}{\pi^5}\right)^{\frac{1}{2}} \beta \omega_1^2 L_0^2 \frac{w_0^3}{V^3} \frac{V^{\frac{5}{2}}}{(\beta z)^{\frac{3}{2}} |\mathbf{V} - \boldsymbol{\beta} z|} \frac{1}{\sin^{\frac{1}{2}} \delta} P\left(\frac{2}{\pi^{\frac{1}{2}}} \frac{\omega_1 L_0}{V \sin \epsilon}\right) Q^{\frac{1}{2}}\left(\frac{2}{\pi^{\frac{1}{2}}} \frac{\omega_1 L_0}{V \sin \epsilon}\right), \tag{6.14}$$

assuming the two average values of $\boldsymbol{\beta} \, . \, \mathbf{k}$ to be nearly the same.

The length and velocity scales of the turbulent motion can be estimated from the well-known relations,

$$\nu_T = u_0 L_u \quad \text{and} \quad \bar{\epsilon} = u_0^3 / L_u,$$

where u_0 is nearly the root-mean-square fluctuation of velocity and L_u is nearly the integral scale. In terms of $\bar{\epsilon}$ and ν_T,

$$u_0 = (\bar{\epsilon} \nu_T)^{\frac{1}{4}} \quad \text{and} \quad L_u = \nu_T^{\frac{3}{4}} \bar{\epsilon}^{-\frac{1}{4}}. \tag{6.15}$$

In these expressions, $\bar{\epsilon}$ is the rate of transfer of energy to heat through the turbulent transfer process and may be rather less than the transfer from the wave-motion. The approximations used are such that the difference may be ignored.

7. Numerical estimates of wave-motion and turbulence

Broadly, the several estimates all show that maximum wave-amplitude occurs either just outside the boundary layer (in reflexion conditions with $\boldsymbol{\beta} \, . \, \mathbf{V}$ negative) or near a height of V/β (in accumulation conditions with $\boldsymbol{\beta} \, . \, \mathbf{V}$ positive). The magnitudes depend on the nature of the boundary layer, on the Brunt–Väisälä frequency and on the elapsed time expressed as a multiple of β^{-1}. The parameters used to describe the disturbance produced by the boundary layer, V, L_0, w_0 and τ_1, have not been measured in the atmosphere and estimates of their values depend on measurements in constant-density layers in the laboratory. Bradshaw (1966) has analysed measurements of the irrotational motion just outside the layer and finds that the convection velocity is approximately $V = 2{\cdot}5u^*$, where u^* is the friction velocity. It is likely that the auto-correlation time-scale

τ_1 is near L_0/u^* and that the root-mean-square displacement velocity is near $\frac{1}{2}u^*$, i.e. $w_0/V = 0{\cdot}2$. With rather more uncertainty, extrapolation to the atmospheric boundary layer indicates that the values,

$$w_0 = 40\,\text{cm}\,\text{sec}^{-1},\quad V = 200\,\text{cm}\,\text{sec}^{-1},\quad L_0 = 200\,\text{m},$$
$$\tau_1 = \tfrac{1}{2}L_0/w_0 = 250\,\text{sec},$$

may be appropriate to a friction velocity of 80 cm sec^{-1}, or a wind of around 13 m sec^{-1}. Assuming these values for w_0, V and L_0, and a characteristic frequency of $\omega_1 = 10^{-2}\,\text{sec}^{-1}$, wave-amplitudes and parameters of wave-produced turbulence have been calculated for the heights of nearly maximum disturbance and, since the value of τ_1 is only an informed guess, for two values of $\omega_1\tau_1$. They are tabulated first as functions of the non-dimensional time βt, and then for $\beta t = 20$, as for a wind-shear of 2 m sec^{-1} km^{-1} and an elapsed time of 10^4 sec or about 3 h. Particle velocities can be found by multiplying the vertical displacements by ω_1, i.e. velocities in cm sec^{-1} are numerically equal to the vertical displacements in metres. The velocities are almost vertical for reflexion components and almost horizontal for accumulation components.

	$\omega_1\tau_1 = 1$		$\omega_1\tau_1 = 2$	
δ (°)	$\langle\zeta^2\rangle/(\beta t)$ (m^2)	$\langle\zeta^2\rangle^{\frac{1}{2}}$ (m)	$\langle\zeta^2\rangle/(\beta t)$ (m^2)	$\langle\zeta^2\rangle^{\frac{1}{2}}$ (m)
0	520	102	700	118
45	410	90	540	104
90	190	62	250	71
135	40	28	55	33
180	—	—	—	—

Note. Calculated from equation (4.13) for $w_0 = 40$ cm sec^{-1}, $V = 200$ cm sec^{-1}, $L_0 = 200$ m and the two indicated values of $\omega_1\tau_1$. The values are for small values of $\beta z/V$ where maximum intensity occurs. The columns of $\langle\zeta^2\rangle^{\frac{1}{2}}$ are for an elapsed time of 10^4 sec or roughly 3 h.

TABLE 1. Vertical displacements from reflexion components.

	$\omega_1\tau_1 = 1$				$\omega_1\tau_1 = 2$			
δ (°)	$\left[t\frac{d}{dt}\langle\zeta^2\rangle\right]^{\frac{1}{2}}$ (m)	$\langle\zeta^2\rangle^{\frac{1}{2}}$ (m)	$\frac{\langle(\partial\zeta/\partial z)^2\rangle^{\frac{1}{2}}}{\beta t}$	$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle^{\frac{1}{2}}$	$\left[t\frac{d}{dt}\langle\zeta^2\rangle\right]^{\frac{1}{2}}$ (m)	$\langle\zeta^2\rangle^{\frac{1}{2}}$ (m)	$\frac{\langle(\partial\zeta/\partial z)^2\rangle^{\frac{1}{2}}}{\beta t}$	$\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle^{\frac{1}{2}}$
5·8	4·4	7·6	0·0²1	0·002	8·8	15	0·002	0·004
45	9·4	16	1·5	0·3	16·5	28	2·5	0·50
90	10·0	17	2·26	0·45	16·0	27	3·3	0·67
135	10·3	18	2·8	0·56	18	31	4·0	0·80
174·2	30	40	7·4	1·48	49	84	10·5	2·1

Note. Calculated from equations (5.3) and (5.8) for $w_0 = 40$ cm sec^{-1}, $V = 200$ cm sec^{-1}, $L_0 = 200$ m and the two indicated values of $\omega_1\tau_1$. The columns of $\langle\zeta^2\rangle^{\frac{1}{2}}$ and $\left\langle\left(\frac{\partial\zeta}{\partial z}\right)^2\right\rangle^{\frac{1}{2}}$ are for an elapsed time of 10^4 sec or roughly 3 h.

TABLE 2. Vertical displacements and gradients from accumulation components for $\beta z/V = 1$.

δ (°)	$\omega_1\tau_1 = 1$: $\frac{\omega_1 \bar{\epsilon}}{\beta}$	$\frac{\omega_1 \nu_T}{\beta}$	$\left(\frac{\omega_1}{\beta}\right)^{\frac{1}{2}} u_0$	$\left(\frac{\omega_1}{\beta}\right)^{\frac{1}{2}} L_u$ (m)	$\omega_1\tau_1 = 2$: $\frac{\omega_1 \bar{\epsilon}}{\beta}$	$\frac{\omega_1 \nu_T}{\beta}$	$\left(\frac{\omega_1}{\beta}\right)^{\frac{1}{2}} u_0$	$\left(\frac{\omega_1}{\beta}\right)^{\frac{1}{2}} L_u$ (m)
5·8	$0{\cdot}0^{3}8$	0·2	0·63	3	0·001	0·8	0·94	8·5
45	0·09	2·9	4·0	7·3	0·28	8·1	6·9	11·7
90	0·10	4·8	4·6	7·8	0·36	10·4	7·8	13·3
135	0·16	8·1	6·0	13·5	0·54	16·4	9·7	16·9
174·2	2·8	180	26·7	67	8·5	370	42	88

Note. Calculated from equations (6.10), (6.14) and (6.15) for $w_0 = 40\ \text{cm sec}^{-1}$, $V = 200\ \text{cm sec}^{-1}$, $L_0 = 200$ m and the two indicated values of $\omega_1\tau_1$. Values of $\bar{\epsilon}$, ν_T and u_0 are in c.g.s. units and of L_u in metres. They refer to the steady state attained *after* the wave-motion has become unstable.

TABLE 3. Characteristics of turbulence generated by wave-accumulation for $\beta z/V = 1$.

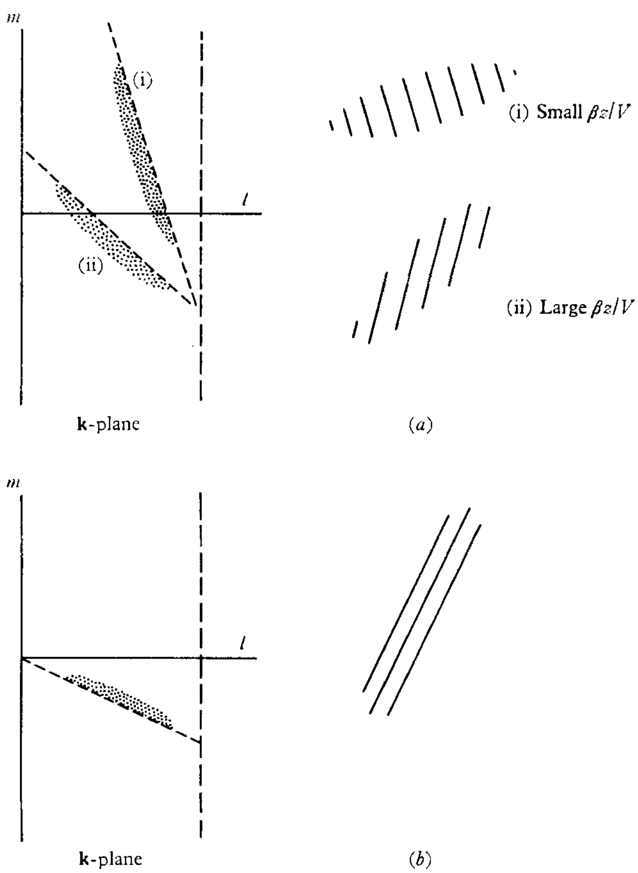

FIGURE 9. General appearance of wave-groups caused by the reflexion process (a) and by the accumulation process (b). The dotted regions in the wave-number plane indicate the dominant wave-numbers, and the lines of the wave-groups indicate crests and troughs of the wave-motion.

From the values in tables 1 and 2, vertical displacements of over 50 m are expected after an elapsed time of 3 h, whatever the angle δ between the directions of the convection velocity and the wind-shear. (The difference between the intensity of reflected waves at small non-dimensional heights and at unit non-dimensional height is not large.) For most relative directions, reflexion components contribute most to the vertical displacements and the wave-pattern at any height is made up of wave-numbers near the reflexion line, i.e. its elements are wave-groups elongated in the direction of the relative wind, $\mathbf{V}-\boldsymbol{\beta}z$, and with an average wave-number of roughly $\omega_1(\mathbf{V}-\boldsymbol{\beta}z)/|\mathbf{V}-\boldsymbol{\beta}z|^2$. For δ near 180°, i.e. wind decreasing with height, the accumulation components are dominant and the elementary wave-groups are elongated still in the direction of the relative wind but with wave-number at right-angles to the wind-direction and of magnitude near $\frac{1}{2}\omega_1/V$. The form of the wave-groups is sketched in figure 9.

Instability of the wave-motion and local production of turbulence requires the attainment of values of $(\partial\zeta/\partial z)^2$ near 1, and table 2 shows that this is likely within a period of hours only if δ exceeds 90°. With a wind-shear of 0·002 sec^{-1}, the turbulence is likely to be rather weak with an energy dissipation of less than 2 cm^2 sec^{-3} and velocity fluctuations less than 20 cm sec^{-1}. The tabulated values are sensitive to the values assumed for w_0/V and for $\omega_1\tau_1$, but the calculations make it hard to believe that the energy-dissipation can exceed 20 cm^2 sec^{-3} and the R.M.S. velocity fluctuation 50 cm sec^{-1}, remembering that in practice convection velocity is not defined precisely enough to assume effective values of $\sin\delta$ less than 0·1. However, the possibility remains that, since the calculated figures are spatial averages, a high degree of inhomogeneity in the distribution of turbulent flow might lead to much higher local values. In view of the highly stable nature of the flow, some inhomogeneity in the distribution of turbulent regions is to be expected. The predicted eddy viscosities are around 5×10^4 cm^2 sec^{-1} for the largest value of $\omega_1\tau_1$ and are far from being negligible, while the integral scales are of the right order of magnitude for clear-air turbulence.

8. Momentum and energy transfer from the boundary layer

The calculation of wave-intensities starts with specified vertical displacements just outside the boundary layer which is assumed capable of maintaining the necessary flux of momentum and energy by the waves. Whether radiated fluxes affect the motion in the boundary layer depends on their magnitudes compared with the Reynolds stresses and rates of energy production in a non-radiating layer. To the wave-packet approximation, the flux of energy leaving the layer by radiation is

$$W = \int k^{-1}(\mathbf{k}\,.\,\mathbf{V})^2(\omega_1^2-l^2V^2)^{\frac{1}{2}}F_0(\mathbf{k})\,d\mathbf{k} \qquad (8.1)$$

(equation 2.14) and, substituting the special form for $F_0(\mathbf{k})$,

$$F_0(k) = \frac{16}{\pi^3}w_0^2L_0^4V^{-2}\frac{l^2V^2\tau_1^2}{1+l^2V^2\tau_1^2}\frac{k^2}{l^2}\exp\left(-\frac{4}{\pi}k^2L_0^2\right)$$

we find
$$W = \frac{32}{\pi^3} w_0^2 L_0^4 \int_{-\infty}^{\infty} \int_0^{\omega_1/V} \frac{l^2 V^2 \tau_1^2}{1+l^2V^2\tau_1^2} (\omega_1^2 - l^2V^2)^{\frac{1}{2}} k \exp\left(-\frac{4}{\pi} k^2 L_0^2\right) dl\, dm \quad (8.2)$$
$$= \frac{8}{\pi^{\frac{3}{2}}} w_0^2 \omega_1 L_0 \left(\frac{4}{\pi}\frac{\omega_1^2 L_0^2}{V^2}\right)^{\frac{3}{2}} \int_0^1 \int_0^{\infty} \frac{\omega_1^2 \tau_1^2 x^2}{1+\omega_1^2\tau_1^2 x^2} (1-x^2)^{\frac{1}{2}} (x^2+y^2)^{\frac{1}{2}} \times \exp\left[-\frac{4}{\pi}\frac{\omega_1^2 L_0^2}{V^2}(x^2+y^2)\right] dx\, dy.$$

For small values of $\omega_1 L_0/V$,
$$W = \frac{2}{\pi} w_0^2 \omega_1^2 L_0^2 V^{-1} \left[\frac{(1+\omega_1^2\tau_1^2)^{\frac{1}{2}} - 1}{\omega_1 \tau_1}\right]^2 \quad (8.3)$$
while, for large values of $\omega_1 L_0/V$,
$$\left.\begin{aligned} W &= w_0^2 \omega_1 L_0 && \text{if} \quad V\tau_1/L_0 \gg 1, \\ &= \frac{3\pi}{16} w_0^2 \omega_1 L_0 (V\tau_1/L_0)^2 && \text{if} \quad V\tau_1/L_0 \ll 1. \end{aligned}\right\} \quad (8.4)$$

If the rate of energy radiation is comparable with the total rate of production of turbulent energy in the boundary layer, it is certain that the turbulent flow will differ from that in an ordinary layer in a neutral environment. Excluding the equilibrium, constant-stress, layer where production and dissipation of turbulent energy are nearly equal, the total rate of energy production is
$$\int_0^{\infty} -\overline{uw}\frac{\partial U}{\partial z}\, dz \approx w_0^2 V \quad (8.5)$$
since w_0^2 appears as a measure of the Reynolds stress and V as a measure of the variation of mean velocity. Both w_0 and V are proportional to the friction velocity. The persistence time τ_1 is characteristic of the turbulent motion and is of order L_0/w_0, i.e. $w_0\tau_1/L_0 \approx 1$. Then equation (8.3) shows that for small values of $\omega_1 L_0/V$,
$$W/(w_0^2 V) = \frac{1}{2\pi} \frac{w_0^2 \tau_1^2}{L_0^2} \frac{V^2}{w_0^2} \left(\frac{\omega_1 L_0}{V}\right)^4 \quad (8.6)$$
indicating that the proportion of energy lost by radiation varies as $(\omega_1 L_0/V)^4$ and is negligible if the scale L_0 is small compared with V/ω_1. The scale is comparable with the thickness of the layer, so the boundary-layer motion is almost unaffected by the radiation if its thickness is substantially less than V/ω_1.

For large values of $\omega_1 L_0/V$, equation (8.4) shows that the energy ratio is
$$\left.\begin{aligned} W/(w_0^2 V) &= \frac{\omega_1 L_0}{V} && \text{if} \quad V\tau_1/L_0 \gg 1, \\ &= \frac{3\pi}{16}\frac{V^2\tau_1^2}{L_0^2}\frac{\omega_1 L_0}{V} && \text{if} \quad V\tau_1/L_0 \ll 1, \end{aligned}\right\} \quad (8.7)$$
and, if the motion in the layer remains unaffected by the radiation, the ratio increases linearly with $\omega_1 L_0/V$, i.e. with the layer thickness. Since all the energy radiated derives from turbulent energy produced in the layer, it is impossible for the ratio to exceed one and the turbulent motion must be completely changed if the layer thickness is to be large compared with V/ω_1. The most likely change is

for τ_1 to become very small compared with L_0/V, equivalent to dominance of wave-like motions which are not really distinguishable from the radiated waves. If we regard the layer thickness as the thickness of characteristically turbulent flow, we must conclude that the maximum possible thickness is about V/ω_1.

The values for L_0 and V used in the previous section were intended to be representative of the earth's boundary layer, and they give $\omega_1 L_0/V = 1$. While the choice of V is only an informed guess, it is very likely that actual values of $\omega_1 L_0/V$ are not far removed from unity and so that the thickness of the layer is usually near the upper limit set by radiation loss of energy. The usual treatment of the boundary layer without buoyancy flux assumes a distribution of eddy viscosity with height and finds a time-independent velocity distribution similar to the Ekman spiral (see, for example, Ellison 1956). The vertical extent is controlled by cancellation of the waves of absolute momentum diffusing upwards from the rotating earth, and the thickness is a multiple of u^*/f (u^* friction velocity, f Coriolis parameter). The present calculations seem to show that energy loss by radiation may exert at least an equal influence on the boundary layer.

The radiated waves carry momentum as well as energy, and a sinusoidal wave of amplitude a and local frequency ω has a vertical flux of momentum in the Ox direction of

$$\tfrac{1}{2}l/k\omega(\omega_1^2-\omega^2)^{\frac{1}{2}}\,aa^*,$$

independent of height but changing sign on reflexion. At low levels, the initial wave 'Reynolds stress' before reflected waves return is

$$\begin{aligned}\tau_\omega(0) &= \int k^{-1}l^2V(\omega_1^2-l^2V^2)^{\frac{1}{2}}F_0(\mathbf{k})\,d\mathbf{k}\\ &= WV^{-1}.\end{aligned} \tag{8.8}$$

After a long time, the average contribution of reflexion components ($\mathbf{k}.\boldsymbol{\beta} < 0$) will be one-half the initial contribution and the low-level stress will be between $\frac{1}{2}W/V$ and W/V. Notice that, if energy-loss by radiation is comparable with total energy production in the turbulent layer, the wave Reynolds stress is comparable with the turbulent stresses and so the top of the turbulent layer is where turbulent stresses are converted to wave stresses. Reflexion and accumulation both prevent the upward transmission of shear stress, and the wave stress becomes small at heights much larger than V/β. The stress gradient is of order $\beta w_0^2/V$ if $\omega_1 L_0/V$ is near one, and could produce an additional component of wind whose ratio to V is about $\beta w_0^2/(fV^2)$. To some extent, the boundary layer may be considered to have a total thickness of about V/β, in which total stress is appreciable and wind velocity differs from the geostrophic values.

9. Concluding remarks

Although the results can be applied to other examples of boundary-layer flow with a stably-stratified environment, the emphasis has been on the atmospheric boundary layer and its interaction with the rest of the troposphere in its normal state of stable stratification. The principal conclusions are:

(i) Internal waves radiated from the earth's boundary layer can lead to vertical displacements of 50 m or more after a growth period of a few hours. Seen from the ground, the wave-patterns travel at the convection velocity, with about four-fifths of the wind-speed at the top of the layer, and, if the vertical displacements cause cloud formation, the wave-clouds travel at this velocity. If wind increases with height, the typical wave-groups are elongated perpendicular to the wave-crests. If it decreases with height, they are elongated along the wave-crests and the group should contain no more than two crests in all.

(ii) Patches of clear-air turbulence are likely to form if wind decreases with height, and their intensity will be greatest where the difference between the wind velocity and the convection velocity of the disturbances is least. Estimates of the intensity and scale suggest the velocity fluctuations are not likely to exceed 50 cm sec^{-1} and that the depth of the patches may be about 50 m. Their presence could lead to effective eddy transfer coefficients in the range 10^4–10^5 $cm^2 sec^{-1}$.

(iii) For a particular boundary layer, the mean square velocities and displacements are proportional to the ratio of vertical wind gradient to the Brunt–Väisälä frequency, i.e. inversely to the square-root of the Richardson number.

(iv) Radiation fluxes of energy and momentum from the boundary layer are sufficient to modify its structure considerably, and it is possible that, in neutral conditions, the earth's boundary layer is nearly as thick as is dynamically possible in the stably-stratified environment. In other words, the radiation of internal waves restricts the layer thickness and the effects are comparable with those of the Coriolis forces.

Wind-driven boundary layers below the ocean surface also have a stable environment and most of the considerations of energy and momentum loss by radiation apply to them. In the atmospheric layer, the thickness set by rotation is close to the limit set by radiation because the ratio of characteristic frequency to Coriolis parameter is nearly 100 (the comparatively large number arises from the large ratio of the effective thickness of a turbulent Ekman layer to the 'scale' u^*/f). It may be a coincidence that the characteristic frequency in the ocean below the surface layer is about 10^{-2} sec^{-1}, nearly the same value as in the atmosphere, but it suggests that the two layers are dynamically similar even allowing for the effects of radiation. A sample of two is far too small for statistical analysis, but it is interesting to speculate whether the general circulation of a fluid on a rotating earth necessarily leads to a stratification determined by the rotation rate.

With reference to the calculations themselves, it should be pointed out that the earlier work on a stratified atmosphere without wind-shear showed that most of the wave-energy appears in the form of wave-components with frequency slightly less than the characteristic frequency of the lower of the two layers of the model, i.e. of the 'troposphere'. The packet approximation cannot be used for these components, and the results are peculiar to the two-layer atmosphere without appreciable wind-shear. That comparable intensities are found is probably due to the rate of radiation of wave-energy being given by the packet approximation to an order of magnitude even for the two-layer atmosphere.

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